Lecture №11 **Probability theory**

INTRODUCTION

*Events* (phenomena) observed by us can be subdivided on the following three kinds: *reliable, impossible and random*.

A reliable (universal) event is an event that necessarily will happen if a certain set of conditions S holds. For example, if a vessel contains water with a normal atmospheric pressure and temperature 20 degrees, the event «water in a vessel is in a liquid state» is reliable. In this example the given atmospheric pressure and temperature of water make the set of conditions S.

An impossible (null) event is an event that certainly will not happen if the set of conditions S holds. For example, the event «water in a vessel is in a rigid state» certainly will not happen if the set of conditions of the previous example holds.

A random event is an event that can either take place, or not to take place for holding the set of conditions S. For example, if a coin is tossed it can land on one of two sides: heads or tails. Therefore, the event «the coin lands on heads» is random. Each random event, in particular an appearance of heads, is the consequence of a functioning very many random reasons (in our example: the force with which the coin is tossed, the form of the coin and many others). It is impossible to take into account an influence on result of all these reasons as their number is very great and laws of their functioning are unknown. Therefore probability theory does not pose the problem to predict that a single event whether or not will happen – it simply is unable to solve this problem.

We have another picture if we consider random events that can multiply be observed for holding the same conditions S, i.e. if the speech goes on mass homogeneous random events. It appears that a rather large number of homogeneous random events independently from their concrete nature are subordinated to definite regularities, namely probability regularities. *Probability theory* studies these regularities. Thus, the subject of probability theory is studying probability regularities of mass homogeneous random events.

Methods of probability theory are widely applied in various branches of natural sciences and techniques: theory of reliability, theory of mass service, theoretical physics, geodesy, astronomy, theory of shooting, theory of mistakes of observation, theory of automatic control, general theory of communication and many others theoretical and applied sciences. Probability theory serves also for a substantiation of mathematical statistics which is in turn used at planning and organizing a manufacture, at analysis of technological processes, and for many other purposes

All life a person has to make decisions: in personal sphere (in which university or college to enter, with whom to communicate; how to study); in public (to attend evenings, theatres, meetings, assemblies, elections); in industrial (determining factors essentially influencing on productivity, quality of materials and etc.); in scientific (promotion and checking scientific hypotheses).

Decision-making usually pursues one of the following purposes: forecasting of a future state of a process (an object); control (i.e. how should to change certain parameters of an object in order that other parameters have taken on a desirable value); an explanation of internal structure of an object.

Usually decision-making is preceded with an analysis of known data (on the basis of previous experience, common sense, intuition, and etc.). Aspiring to see and prove regularities in uncertain processes, the mankind has developed the whole arsenal of methods which refer to as mathematical statistics (applied statistics or data analysis).

Mathematical statistics is a section of mathematics in which mathematical methods of ordering, processing and using statistical data for scientific and practical conclusions are studied.

Abraham Wald (1902-1950) spoke that «mathematical statistics is theory of decision-making in conditions of uncertainty».

In essence mathematical statistics gives a unique, mathematically proved apparatus for solving problems of control and forecasting at absence of obvious regularities (presence of randomness) in investigated processes.

*It is wonderful that the science begun with consideration of gambles was fated to become the major object of human knowledge* …

Pierre-Simon Laplace (1749-1827).

## **1. Basic concepts of probability theory**

Hereinafter, instead of speaking «the set of conditions *S* holds» we shall speak briefly: «***the trial has been made***». Thus, an event will be considered as a result of a trial.

*Example*. A shooter shoots in a target subdivided into four areas. One shot is the trial. Hit in a certain area of the target is an event.

*Example*. There are colour balls in an urn. One takes at random one ball from the urn. An extracting a ball from the urn is the trial. An appearance of a ball of a certain colour is an event.

Events are *incompatible* if an appearance of one of them excludes an appearance of other events in the same trial. Otherwise, they are *compatible*.

*Example*. A detail is extracted at random from a box with details. An appearance of a standard detail excludes an appearance of a non-standard detail. The events «a standard detail has appeared» and «a non-standard detail has appeared» are incompatible.

*Example.* A coin is tossed. An appearance of «heads» excludes an appearance of «tails». The events «heads have appeared» («the coin lands on heads») and «tails have appeared» («the coin lands on tails») are incompatible.

For example, a landing two different prizes under only one ticket of a lottery are incompatible events, and a landing the same prizes under two tickets are compatible events. Obtaining marks «excellent», «good» and «satisfactory» by a student at an exam in one discipline are incompatible events and an obtaining the same marks at exams in three disciplines are compatible events.

Some events form a *complete group* if in result of a trial at least one of them will appear. In other words, the appearance of at least one of events of a complete group is a reliable event.

In particular, if events forming a complete group are pairwise incompatible then in result of a trial one and only one of these events will appear.

*Example*. Two tickets of a money-thing lottery have been bought. One necessarily will happen one and only one from the following events: «a landing a prize on the first ticket and a non-landing a prize on the second», «a landing a prize on both tickets», «a non-landing a prize on the first ticket and a landing a prize on the second», «a non-landing a prize on both tickets». These events form a complete group of pairwise incompatible events.

*Example*. A shooter has made one shot in a target. One necessarily will happen one from the following two events: hit, miss. These two incompatible events form a complete group.

Events are *equally possible* if there is reason to consider that none of them is more possible (probable) than other.

*Example*. An appearance of heads and an appearance of tails at tossing a coin are equally possible events. Appearances of «one», «two», «three», «four», «five» or «six» on a tossed die are equally possible events.

Several events are *uniquely possible* if at least one of them will necessarily happen as a result of a trial. For example, the events consisting in that a family with two children has: *A* – «two boys», *B* – «one boy and one girl» and *C* – «two girls» are uniquely possible.

**Classical definition of probability**

*Example*. Let an urn contain 6 identical, carefully shuffled balls, and 2 of them are red, 3 – blue and 1 – white. Obviously, the possibility to take out at random from the urn a colour ball (i.e. red or blue) is more than the possibility to extract a white ball.

Whether it is possible to describe this possibility by number? It appears it is possible. This number is said to be the probability of an event (appearance of a colour ball). Thus, the probability is the number describing the degree of possibility of an appearance of an event.

Let the event *A* be an appearance of a colour ball. We call each of possible results of a trial (the trial is an extracting a ball from the urn) by *elementary event*. We denote elementary events by ω1, ω2, ω3 and et cetera. In our example the following 6 elementary events are possible: ω1 – the white ball has appeared; ω2, ω3 – a red ball has appeared; ω4, ω5, ω6 – a blue ball has appeared. These events form a complete group of pairwise incompatible events (it necessarily will be appeared only one ball) and they are equally possible (a ball is randomly extracted; the balls are identical and carefully shuffled).

We call those elementary events in which the event interesting for us occurs, as *favorable* to this event. In our example the following 5 events favor to the event *A* (appearance of a colour ball): ω2, ω3, ω4, ω5, ω6. In this sense the event *A* is subdivided on some elementary events; an elementary event is not subdivided into other events. It is the distinction between the event *A* and an elementary event.

The ratio of the number of favorable to the event *A* elementary events to their total number is said to be *the probability of the event A* and it is denoted by *P(A)*. In the considered example we have 6 elementary events; 5 of them favor to the event *А*. Therefore, the probability that the taken ball will be colour is equal to *P(A) = 5/6*. This number gives such a quantitative estimation of the degree of possibility of an appearance of a colour ball which we wanted to find.

*The probability* of the event *A* is the ratio of the number of favorable elementary events for this event to their total number of all equally possible incompatible elementary events forming a complete group.

Thus, the probability of the event *A* is determined by the formula:



where *m* is the number of elementary events favorable to *A*; *n* is the number of all possible elementary events of a trial. Here we suppose that elementary events are incompatible, equally possible and form a complete group.

The definition of probability implies the following its properties:

*Property 1.* The probability of a reliable event is equal to 1, .

In fact, if an event is reliable then each elementary event of a trial favors to the event. In this case *m = n* and consequently .

*Property 2.* The probability of an impossible event is equal to 0, .

Indeed, if an event is impossible then none of elementary events of a trial favors to the event. In this case *m = 0* and consequently .

*Property 3*. The probability of a random event is the positive number between 0 and 1, .

In fact, a random event is favored only part of the total number of elementary events of a trial. In this case *0 < m < n*; then *0 < m/n < 1* and consequently *0 < P(A) < 1*.

Thus, the probability of an arbitrary event *A* satisfies the double inequality:



**Relative frequency**

*The relative frequency (statistical probability)* of an event is the ratio of the number of trials, in which the event has appeared, to the total number of actually made trials.

Thus, the relative frequency of the event *A* is defined by the formula:



where *m* is the number of appearances of the event, *n* is the total number of trials.

Comparing the definitions of probability and relative frequency, we conclude: the definition of probability does not demand that the trials should be made actually; the definition of relative frequency assumes that the trials were made actually. In other words, *the probability is calculated before an experiment, and the relative frequency – after an experiment*.

*Example*. The quality department has detected 3 non-standard details in a group consisting of 80 randomly selected details. The relative frequency of appearance of non-standard details is *W(A) = 3/80*.

*Example*. There have been made 24 shots in a target, and 19 hits were registered. The relative frequency of hit in the target is *W(A) = 19/24*.

The long observations have shown that if experiments are made in identical conditions, in each of which the number of trials is rather great, the relative frequency has a stability property. This property is that for different experiments the relative frequency is changed a little (the less changes, the more trials were made), oscillating about some constant number. There was found out that this constant number is the probability of appearance of the event.

Thus, if the relative frequency is established by a practical experiment, the obtained number can be accepted for approximate value of probability.

**Geometric probabilities**

To overcome defect of the classical definition of probability consisting that it is inapplicable to trials with infinite number of events (outcomes) enter geometric probabilities – the probability of hit of a point in area (segment, part of a plane and etc.).

Let a segment *l* be a part of a segment *L*. A point is set (thrown) at random in the segment *L*. It means that the following suppositions hold: the thrown point can appear in any point of the segment *L*, the probability of hit of the point in the segment *l* is proportional to the length of this segment and does not depend on its disposition concerning the segment *L*. In these suppositions the probability of hit of the point in the segment *l* is determined by the equality

**P = the length of *l* / the length of L**

*Example*. A point *B(x)* is thrown at random in a segment *OA* of the length *L* of the numeric axis *Ox*. Find the probability that the smaller of the segments *OB* and *BA* has the length more than *L/3*. It is assumed that the probability of hit of a point in the segment is proportional to the length of the segment and does not depend on its disposition on the numeric axis.

*Solution*: Let's divide the segment *OA* by points *C* and *D* on three equal parts. The request of the problem will be executed if the point *B(x)* will hit in the segment *CD* of the length *L/3*. The required probability *P = (L/3)/L = 1/3*.

Let a flat figure *g* be a part of a flat figure *G*. A point is thrown at random in the figure *G*. It means that the following suppositions hold: the thrown point can appear in any point of the figure *G*, the probability of hit of the thrown point in the figure *g* is proportional to the area of this figure and does not depend on both its disposition concerning the figure *G* and the form of *g*. In these suppositions the probability of hit of the point in the figure *g* is determined by the equality

**P = the area of g / the area of G**

*Example*. Two concentric circles of which the radiuses are 5 and 10 cm respectively are drawn on the plane. Find the probability that the point thrown at random in the large circle will hit in the ring formed by the constructed circles. It is assumed that the probability of hit of a point in a flat figure is proportional to the area of this figure and does not depend on its disposition concerning the large circle.

*Solution*: The area of the ring (the figure g) . 

The area of the large circle (the figure G) 

The required probability 

**Glossary**

**probability theory** – теория вероятностей

**reliable event** – достоверное событие

**random event** – случайное событие

**vessel** – сосуд; **trial (experiment)** – испытание (опыт, эксперимент)

**urn** – урна; **heads or tails?** – орел или решка?

**at random** – наудачу; **to land a prize** – получить приз

**complete group of events** – полная группа событий

**equally possible events** – равновозможные события

**uniquely possible events** – единственно возможные события

**ace** – очко (при игре в кости); **die** – кость (игральная)

**dice** – игра в кости, кости; **hit** – попадание; **miss** – промах

**to shuffle** – перемешивать; **relative frequency** – относительная частота

**favorable case** – благоприятствующий (благоприятный) случай

**mass homogeneous events** – массовые однородные события

**Basic formulas of combinatorial analysis**

Here is a typical problem of interest involving probability. A communication system is to consist of *n* seemingly identical antennas that are to be lined up in a linear order. The resulting system will then be able to receive all incoming signals – and will be called *functional* – as long as no two consecutive antennas are defective. If it turns out that exactly *m* of the *n* antennas are defective, what is the probability that the resulting system will be functional? For instance, in the special case where *n* = 4 and *m* = 2 there are 6 possible system configurations – namely,

0110, 0101, 1010, 0011, 1001, 1100

where 1 means that the antenna is working and 0 that it is defective. As the resulting system will be functional in the first 3 arrangements and not functional in the remaining 3, it seems reasonable to take 3/6 = 1/2 as the desired probability. In the case of general *n* and *m*, we could compute the probability that the system is functional in a similar fashion. That is, we could count the number of configurations that result in the system being functional and then divide by the total number of all possible configurations.

From the preceding we see that it would be useful to have an effective method for counting the number of ways that things can occur. In fact, many problems in probability theory can be solved simply by counting the number of different ways that a determinate event can occur. The mathematical theory of counting is formally known as *combinatorial analysis*.

Combinatorial analysis studies quantities of ordered sets subordinate to determinate conditions, which can be made of elements, indifferent of a nature, of a given finite set.

***Permutations*** are ordered sets consisting of the same *n* different elements and distinguishing only by the order of their disposition (location).

The number of all possible permutations 

*Example*. How many three-place numbers can be made of the digits 1, 2, 3 if each digit is included into the image of a number only once?

*Solution: P3 = 3! =* 123*= 6.*

*Allocations* are ordered sets composed of *n* different elements on *m* elements, which are differed by either structure of elements or their order.

The number of all possible allocations



*Example*. How many signals is it possible to make of 6 flags of different colour taken on 2?

*Solution:* 

*Combinations* are ordered sets composed of *n* different elements on *m* elements that are differed by at least one element.

The number of all possible combinations 

*Example*. How many ways are there to choose two details from a box containing 10 details?

*Solution*: The required number of ways



Observe that the numbers of allocations, permutations and combinations are connected by the equality



*Remark.* It was above supposed that all *n* elements are different. If some elements are repeated then in this case *ordered sets with repetitions* are calculated by other formulas. For example, if there are *n1* elements of one kind, *n2* elements of other kind and et cetera among *n* elements then *the number of permutations with repetitions*



where *n1 + n2 + … = n*.

*Example.* How many seven-place numbers consisting of the digits 4, 5 and 6 are there in which the digit 4 is repeated 3 times, and the digits 5 and 6 – 2 times?

*Solution:* Each seven-place number differs from other by the order of consecution of the digits (so that *n*1 = 3, *n*2 = 2, *n*3 = 2, and their sum is equal to 7), i.e. is the permutation with repetitions of 7 elements:



If in allocations (combinations) of *n* elements on *m* some of elements (or all) can appear identical, such allocations (combinations) are said to be *allocations (combinations) with repetitions* of *n* elements on *m*. For example, allocations with repetitions of 5 elements *a, b, c, d, e* on 3 are *abc, cba, bcd, cdb, bbe, ebb, beb, ddd* and et cetera; combinations with repetitions – *abc, bcd, bbe, ddd* and et cetera.

The number of allocations with repetitions of *n* elements on *m* is



The number of combinations with repetitions of *n* elements on *m* is



*Example.* 10 films participate in a competition on 5 nominations. How many variants of distribution of prizes are there, if on each nomination are established:

a) different prizes; b) identical prizes?

*Solution:* a) Each of variants of distribution of prizes represents a combination of 5 films from 10 that differs from other combinations by both the structure of elements and their order on nominations so that the same films can be repeated some times (a film can receive prizes on both one nomination and some nominations), i.e. it represents the allocations with repetitions of 10 elements on 5:



b) If identical prizes are established on each nomination then the order of following the films in a combination of 5 prizewinners doesn’t have any significance. Therefore, the number of variants of distribution of prizes represents the number of combinations with repetitions of 10 elements on 5:



The following rules are used for solving problems of combinatorial analysis:

*Sum rule.* If some object *A* can be chosen from the set of objects by *m* ways, and another object *B* can be chosen by *n* ways, then we can choose either *A* or *B* by *m + n* ways.

*Example.* There are 300 details in a box. It is known that 150 of them are details of the first kind, 120 – the second kind, and the rest – the third kind. How many ways of extracting a detail of the first or the second kind from the box are there?

*Solution:* A detail of the first kind can be extracted by *n*1 = 150 ways, of the second kind – by *n*2 = 120 ways. By the sum rule there are *n1 + n2 = 150 + 120 = 270* ways of extracting a detail of the first or the second kind.

*Product rule*. If an object *A* can be chosen from the set of objects by *m* ways and after every such choice an object *B* can be chosen by *n* ways then the pair of the objects (*A, B*) in this order can be chosen by *mn* ways.

*Example.* There are 30 students in a group. It is necessary to choose a leader, its deputy and head of professional committee. How many ways of choosing them are there?

*Solution:* A leader can be chosen from any of 30 students, its deputy – from any of the rest 29 students, and head of professional committee – from any of the rest 28 students, i.e. *n*1 = 30, *n*2 = 29, *n*3 = 28. By the product rule the total number of ways is *n*1 \* *n*2 \* *n*3 = 30 \* 29 \* 28 = 24360.

**Operations over events**

We enter operations over events: sum, product and negation.

*The sum* of two events *A* and *B* is such third event *A + B* which consists in appearance of at least one of these events, i.e. *A* or *B*. If *A* and *B* are compatible

events then their sum *A + B* means appearance of either the event *A*, or the event

*B*, or both *A* and *B*. If *A* and *B* are incompatible events then their sum *A + B* means appearance of either the event *A*, or the event *B*. For example, if two shots are made by a gun and *A* is hit at the first shot, *B* is hit at the second shot then *A + B* is either hit at the first shot or hit at the second shot, or two hits. The event *A + B + C* consists of appearance of one of the following events: *A; B; C;* both *A* and *B;* both *A* and *C;* both *B* and *C;* all the events *A, B* and *C*.

*The product* of two events *A* and *B* is such third event *AB* which consists in simultaneous appearance of the events *A* and *B*. If *A* and *B* are incompatible events then their product *AB* is an impossible event.

*The negation* of an event *A* is the event *A* (not *A*) which consists in nonappearance

of the event *A*. Observe that *A* + *A* is a reliable event, and *A* *A* is an impossible event.

*Example.* A winner of a competition is rewarded: by a prize (the event *А*), a money premium (the event *В*), a medal (the event *С*).

What do the following events represent: a) *A + B*; b) *ABC*; c) *ACB* ?

*Solution:* a) The event *A + B* consists in rewarding the winner by a prize, or a money premium, or simultaneously both a prize and a money premium.

b) The event *ABC* consists in rewarding the winner by a prize, a money premium and a medal simultaneously.

c) The event *ACB* consists in rewarding the winner by both a prize and a medal

simultaneously without giving a money premium.

**Glossary**

**seemingly –** на вид, по-видимому; **arrangement –** расположение

**allocation** – размещение; **combination** – сочетание

**permutation** – перестановка; **three-place number** – трехзначное число

**consecution** – следование; **repetition** – повторение

**to extract** – извлекать; **simultaneous** – одновременный

negation – отрицание